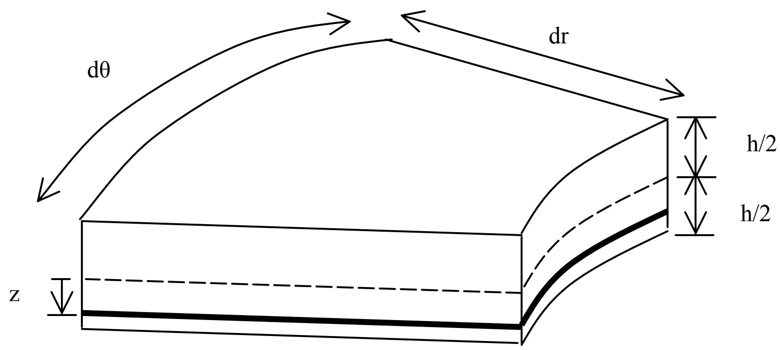
BENDING OF CIRCULAR PLATES UNDER SYMMETRICAL DISTRIBUTED LATERAL LOAD (SMALL DEFLECTION THEORY)

1. ASSUMPTIONS

a. There is no deformation in the mid-plane.
b. Points lying initially on a normal to the mid-plane, remain on the normal during bending.
c. Normal stresses in the thickness direction can be neglected.

2. MOMENT-DEFLECTION RELATIONSHIPS:-



At a depth z below the neutral surface, the strains in the r and θ directions of a layer are (recalling Beam Theory):

$$\varepsilon_r = \frac{z}{R_r} \qquad (1)$$

$$\varepsilon_\theta = \frac{z}{R_\theta} \qquad (2)$$

The stress-strain relationships are:

$$\varepsilon_r = \frac{\sigma_r}{E} - \frac{\upsilon\sigma_\theta}{E} \qquad (3)$$

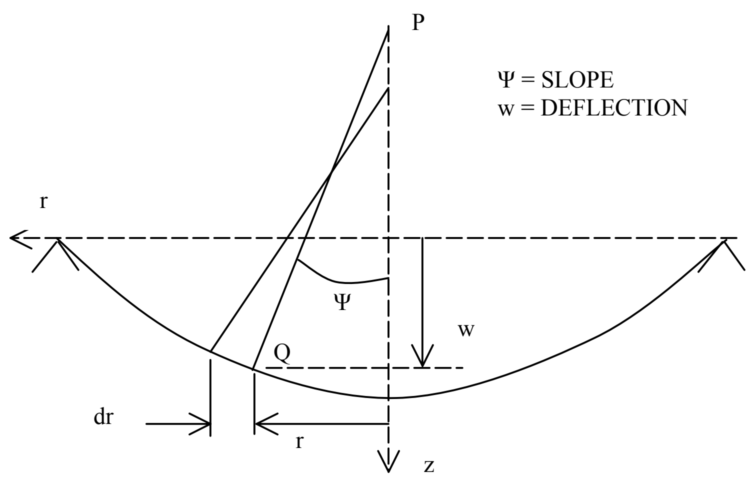
$$\varepsilon_\theta = \frac{\sigma_\theta}{E} - \frac{\upsilon\sigma_r}{E} \qquad (4)$$

Substitute (3) (4) in (1), (2)

$$\sigma_r = \frac{Ez}{1-\upsilon^2}\left(\frac{1}{R_r} + \frac{\upsilon}{R_\theta}\right) \qquad (5)$$

$$\sigma_\theta = \frac{Ez}{1-\upsilon^2}\left(\frac{1}{R_\theta} + \frac{\upsilon}{R_r}\right) \qquad (6)$$

Hence the bending stresses are proportional to the distance from the neutral surface, and are a function of the plate curvatures. Consider the case of a symmetrically distributed lateral load (i.e. an axisymmetric load).



The plate curvature in a diametral section r-z is R_r and (as per Beam Theory) is given by,

$$\frac{1}{R_r} = -\frac{d^2w}{dr^2} \qquad (7)$$

For small values of w, the slope at any point Ψ is given by,

$$\Psi = \frac{-dw}{dr} \qquad (8)$$

For axisymmetric loading, R_θ is in a plane perpendicular to r-z such that PQ forms a conical surface (i.e. for the same r anywhere on the plate, the slope Ψ is constant therefore not a function of θ). Hence,

$$\frac{1}{R_\theta} = \frac{\Psi}{r} \qquad (9)$$

This has the important effect of subsequently requiring total rather than partial derivatives.

Now by definition,

$$M_r d\theta = \int_{-h/2}^{h/2} z\sigma_r d\theta dz \tag{10}$$

$$M_\theta dr = \int_{-h/2}^{h/2} z\sigma_\theta dr dz \tag{11}$$

Substitute (5), (6) in (10), (11) we get,

$$M_r = \frac{Eh^3}{12(1-\upsilon^2)}\left(\frac{1}{R_r} + \frac{\upsilon}{R_\theta}\right) \tag{12}$$

$$M_\theta = \frac{Eh^3}{12(1-\upsilon^2)}\left(\frac{1}{R_\theta} + \frac{\upsilon}{R_r}\right) \tag{13}$$

Let $K = Eh^3/[12(1-v^2)]$ which is called the “flexural rigidity”.

Substitute (7), (9) in (12), (13)

$$M_r = K\left(\frac{d\Psi}{dr} + \upsilon\frac{\Psi}{r}\right) \tag{14}$$

$$M_\theta = K\left(\frac{\Psi}{r} + \upsilon\frac{d\Psi}{dr}\right) \tag{15}$$

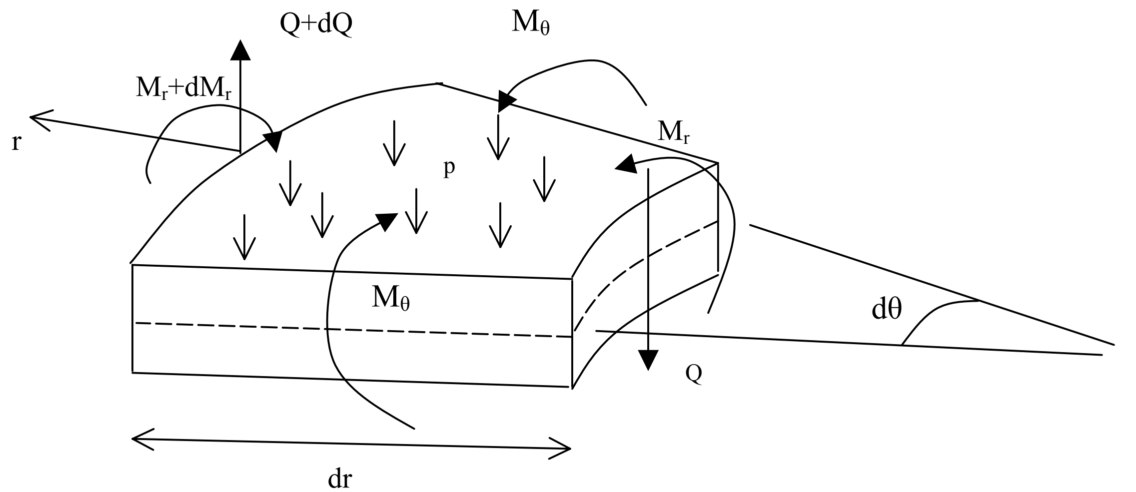
Note that M_r and M_θ are per unit length.

Substitute (8) in (14), (15),

$$M_r = -K\left(\frac{d^2w}{dr^2} + \frac{\upsilon}{r}\frac{dw}{dr}\right) \tag{16}$$

$$M_\theta = -K\left(\frac{1}{r}\frac{dw}{dr} + \upsilon\frac{d^2w}{dr^2}\right) \qquad (17)$$

3. LOAD-SHEAR FORCE – MOMENT RELATIONSHIPS:-



An element of the plate is shown above under a load "p" per unit area which is a function of r only. Q is the shear force per unit length. Notice that for the case of axisymmetric loading, M_θ is constant for all points at r. Hence there is no shear force on those sides. This has the important effect of uncoupling the equilibrium equations.

For vertical equilibrium:

$$Qrd\theta + prdrd\theta - (Q + \frac{dQ}{dr}dr)(r + dr)d\theta = 0$$

$$\frac{dQ}{dr} + \frac{Q}{r} = p \qquad (18)$$

For moment equilibrium:

$$(M_r + \frac{dM_r}{dr}dr)(r + dr)d\theta - M_r rd\theta - 2M_\theta dr\sin\frac{d\theta}{2} + Qrd\theta dr = 0$$

$$r\frac{dM_r}{dr} + M_r - M_\theta + Qr = 0 \qquad (19)$$

4. DEFLECTION-SLOPE-LOAD RELATIONSHIPS:-

Substitute (14), (15) in (19) we get,

$$\frac{d^2\Psi}{dr^2}+\frac{1}{r}\frac{d\Psi}{dr}-\frac{\Psi}{r^2}=-\frac{Q}{K} \tag{20}$$

Substitute (16), (17) in (19) we get,

$$\frac{d^3w}{dr^3}+\frac{1}{r}\frac{d^2w}{dr^2}-\frac{1}{r^2}\frac{dw}{dr}=\frac{Q}{K} \tag{21}$$

From (20),

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\Psi)\right]=-\frac{Q}{K} \tag{22}$$

From (21),

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\frac{dw}{dr})\right]=\frac{Q}{K} \tag{23}$$

From (18),

$$rdQ+Qdr=prdr$$

$$d(Qr)=prdr$$

$$Qr=\int_0^r prdr \tag{24}$$

Substitute (24) in (23),

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\frac{dw}{dr})\right]=\frac{1}{K}\int_0^r prdr \tag{25}$$

5. SOLUTIONS FOR p= CONSTANT :-

When p = constant, the load on the plate is the same at all points on the surface of the plate.

From (25),

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\frac{dw}{dr})\right] = \frac{pr}{2K} \tag{26}$$

Integrating,

$$\frac{1}{r}\frac{d}{dr}(r\frac{dw}{dr}) = \frac{pr^2}{4K} + C_1$$

Multiplying both sides by r and integrating,

$$r\frac{dw}{dr} = \frac{pr^4}{16K} + \frac{C_1 r^2}{2} + C_2$$

Hence,

$$\frac{dw}{dr} = \frac{pr^3}{16K} + \frac{C_1 r}{2} + \frac{C_2}{r} \tag{27}$$

Integrating we get,

$$w = \frac{pr^4}{64K} + \frac{C_1 r^2}{4} + C_2 \ln r + C_3 \tag{28}$$

CASE A. CLAMPED EDGES

For a plate of radius “a” and clamped edges, the boundary conditions are:

(dw/dr) = 0 @ r = 0
(dw/dr) = 0 @ r = a

Hence $C_2 = 0$, and therefore,

$$0 = \frac{pa^3}{16K} + \frac{C_1 a}{2}$$

$$C_1 = \frac{-pa^2}{8K}$$

Since w = 0 @ r = a,

$$0 = \frac{pa^4}{64K} - \frac{pa^4}{32K} + C_3$$

$$C_3 = \frac{pa^4}{64K}$$

Substitute in (28),

$$w = \frac{p}{64K}\left(a^2 - r^2\right)^2 \qquad (29)$$

Maximum Deflection:

The maximum deflection occurs at the plate center (i.e. r=0), therefore

$$w_{MAX} = \frac{pa^4}{64K} \qquad (30)$$

Moment values:

From (16), (17) and (27)

$$M_r = \frac{p}{16}\left[a^2(1+\upsilon) - r^2(3+\upsilon)\right]$$

$$M_\theta = \frac{p}{16}\left[a^2(1+\upsilon) - r^2(1+3\upsilon)\right]$$

At the plate’s edges r = a, hence

$$M_r = \frac{-pa^2}{8} \qquad (31)$$

$$M_\theta = \frac{-\upsilon pa^2}{8}$$

At the plate's center r = 0, hence

$$M_r = M_\theta = \frac{pa^2}{16}(1+\upsilon)$$

Stress Values:

From (5) and (12),

$$\sigma_r = \frac{12Mz}{h^3}$$

At the surface, z = ± h/2 therefore

$$\sigma_{r,MAX} = \pm\frac{6M_{r,MAX}}{h^2}$$

This occurs at the plate's edge. Therefore from (31),

$$\sigma_{r,MAX} = \pm\frac{3pa^2}{4h^2}$$